ON A REFINEMENT OF THE CLASSICAL THEORY OF BENDING OF CIRCULAR PLATES

(OB UTOCHNENII KLASSICHESKOI TEORII IZGIBA KRUGLYKH PLASTINOK)

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A.V. KOLOS
(Moscow)

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A method proposed in [1] for refining the classical theory of bending of plates is considered in application to the problem of a circular plate and an infinite plate with a circular hole.

1. The problems formulated are solved in cylindrical coordinates. The origin of the cylindrical system is chosen at the center of the circumference of the plate in its middle plane, and the z-axis is perpendicular to this plane. The thickness of the plate is $2h$, and its radius is R. The state of stress in the plate is the sum of the basic state of stress, as it was called in [1], and two auxiliary states of stress which decay rapidly with distance from the edge.

In the integration of the equations of the three-dimensional problem in elasticity theory, all stresses and displacements Q are written in the form

$$Q = h^{-q} \sum_{s=1}^{s=S} h^{s-1} Q^{(s)} \tag{1.1}$$

where q is an integer which in each iteration process for the various stresses and displacements is chosen in the same manner as in [1].

For the basic state of stress which is constructed by means of the basic iteration process, q has the values

$$\begin{gathered} q = 2 \quad \text{for } \sigma_r, \sigma_\theta, \tau_{r\theta}, \qquad q = 1 \quad \text{for } \tau_{rz}, \tau_{\theta z} \\ q = 0 \quad \text{for } \sigma_z, \qquad q = 2 \quad \text{for } u, v, \qquad q = 3 \quad \text{for } w \end{gathered} \tag{1.2}$$

The system of equations [1] which is satisfied at each approximation by the stresses and displacements $Q^{(s)}$ of this state of stress has in the present case the following form in cylindrical coordinates:

$$\frac{\partial \sigma_r^{(s)}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}^{(s)}}{\partial \theta} + \frac{\partial \tau_{rz}^{(s)}}{\partial \zeta} + \frac{\sigma_r^{(s)} - \sigma_\theta^{(s)}}{r} = 0 \tag{1.3}$$

$$\frac{\partial \tau_{r\theta}^{(s)}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta^{(s)}}{\partial \theta} + \frac{\partial \tau_{\theta z}^{(s)}}{\partial \zeta} + \frac{2\tau_{r\theta}^{(s)}}{r} = 0, \qquad \frac{\partial \tau_{rz}^{(s)}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}^{(s)}}{\partial \theta} + \frac{\partial \sigma_z^{(s)}}{\partial \zeta} + \frac{\tau_{rz}^{(s)}}{r} = 0$$

$$E\frac{\partial u^{(s)}}{\partial r} = \sigma_r^{(s)} - \nu\,(\sigma_\theta^{(s)} + \sigma_z^{(s-2)}), \quad E\left(\frac{1}{r}\frac{\partial v^{(s)}}{\partial \theta} + \frac{u^{(s)}}{r}\right) = \sigma_\theta^{(s)} - \nu\,(\sigma_r^{(s)} + \sigma_z^{(s-2)}) \quad (1.3)$$

cont.

$$E\,\frac{\partial w^{(s)}}{\partial \zeta} = \sigma_z^{(s-4)} - \nu\,(\sigma_r^{(s-2)} + \sigma_\theta^{(s-2)}), \quad E\left(\frac{1}{r}\frac{\partial u^{(s)}}{\partial \theta} + \frac{\partial v^{(s)}}{\partial r} - \frac{v^{(s)}}{r}\right) = 2\,(1+\nu)\,\tau_{r\theta}^{(s)}$$

$$E\left(\frac{\partial v^{(s)}}{\partial \zeta} + \frac{1}{r}\frac{\partial w^{(s)}}{\partial \theta}\right) = 2\,(1+\nu)\,\tau_{\theta z}^{(s-2)}, \quad E\left(\frac{\partial w^{(s)}}{\partial r} + \frac{\partial u^{(s)}}{\partial \zeta}\right) = 2\,(1+\nu)\,\tau_{rz}^{(s-2)}$$

The solution of the system (1.3) is represented as the sum $Q_i^{(s)} + Q^{*\,(s)}$, where $Q_i^{(s)}$ is the solution of the homogeneous system which we obtain by discarding terms in Equations (1.3) with the superscripts $(s-2)$ and $(s-4)$, and $Q^{*\,(s)}$ is any particular solution of the inhomogeneous system (1.3), in which the quantities with superscripts $(s-2)$ and $(s-4)$ are treated as known. We easily obtain the solution $Q_i^{(s)}$ in the form

$$w_i^{(s)} = w_0^{(s)}(r,\theta), \qquad u_i^{(s)} = \zeta u_1^{(s)}(r,\theta), \qquad v_i^{(s)} = \zeta v_1^{(s)}(r,\theta)$$

$$\sigma_{ri}^{(s)} = \zeta\sigma_{r1}^{(s)}(r,\theta), \qquad \sigma_{\theta i}^{(s)} = \zeta\sigma_{\theta 1}^{(s)}(r,\theta)$$

$$\tau_{r\theta i}^{(s)} = \zeta\tau_{r\theta 1}^{(s)}(r,\theta), \qquad \tau_{rzi}^{(s)} = \zeta^2\tau_{rz2}^{(s)}(r,\theta) + \tau_{rz0}^{(s)}(r,\theta) \qquad (1.4)$$

$$\tau_{\theta zi}^{(s)} = \zeta^2\tau_{\theta z2}^{(s)}(r,\theta) + \tau_{\theta z0}^{(s)}(r,\theta), \qquad \sigma_{zi}^{(s)} = \zeta^3\sigma_{z3}^{(s)}(r,\theta) + \zeta\sigma_{z1}^{(s)}(r,\theta)$$

(where only those powers of ζ which correspond to the problem of bending are retained).

We also have the relations

$$u_1^{(s)} = -\frac{\partial w_0^{(s)}}{\partial r}, \qquad v_1^{(s)} = -\frac{1}{r}\frac{\partial w_0^{(s)}}{\partial \theta}$$

$$\sigma_{r1}^{(s)} = -\frac{E}{1-\nu^2}\left[\frac{\partial^2 w_0^{(s)}}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w_0^{(s)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0^{(s)}}{\partial \theta^2}\right)\right] \qquad (1.5)$$

$$\sigma_{\theta 1}^{(s)} = -\frac{E}{1-\nu^2}\left[\frac{1}{r}\frac{\partial w_0^{(s)}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w_0^{(s)}}{\partial \theta^2} + \nu\frac{\partial^2 w_0^{(s)}}{\partial r^2}\right]$$

$$\tau_{r\theta 1}^{(s)} = -\frac{E}{1+\nu}\left[\frac{1}{r}\frac{\partial^2 w_0^{(s)}}{\partial r\,\partial \theta} - \frac{1}{r^2}\frac{\partial w_0^{(s)}}{\partial \theta}\right]$$

$$\tau_{rz2}^{(s)} = \frac{E}{2(1-\nu^2)}\frac{\partial}{\partial r}(\Delta w_0^{(s)}), \qquad \tau_{\theta z2}^{(s)} = \frac{E}{2(1-\nu^2)}\frac{1}{r}\frac{\partial}{\partial \theta}(\Delta w_0^{(s)})$$

$$\sigma_{z3}^{(s)} = -\frac{E}{6(1-\nu^2)}\Delta\Delta w_0^{(s)}, \qquad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

The quantities $Q^{*\,(s)}$ are expressed in an elementary way in quadratures with respect to ζ. Recursion formulas for them are similar to the corresponding formulas in Cartesian coordinates. As in Cartesian coordinates, $Q^{*\,(s)} \equiv 0$ for $s = 1, 2$.

Applying the boundary conditions on the upper and lower planes of the plate

$$\sigma_z^{(1)} = \pm {}^1\!/_2\, p\,(r,\theta), \quad \sigma_z^{(s)} = 0 \;\; (s \geqslant 2), \quad \tau_{rz}^{(s)} = \tau_{\theta z}^{(s)} = 0 \;\; (s \geqslant 1) \quad \text{for } \zeta = \pm 1 \qquad (1.6)$$

we obtain

$$\sigma_{z3}^{(1)} = -{}^1/_4\, p, \quad \sigma_{z1}^{(1)} = {}^3/_4\, p, \quad \sigma_{z3}^{(2)} = \sigma_{z1}^{(2)} = 0$$

$$\sigma_{z3}^{(s)} = {}^1/_2 \left[\sigma_z^{*\,(s)} - \frac{\partial \sigma_z^{*\,(s)}}{\partial \zeta} \right]_{\zeta=1} \qquad (s > 2)$$

$$\sigma_{z1}^{(s)} = -{}^1/_2 \left[3\sigma_z^{*\,(s)} - \frac{\partial \sigma_z^{*\,(s)}}{\partial \zeta} \right]_{\zeta=1} \qquad (s > 2) \tag{1.7}$$

$$\tau_{rz0}^{(1)} = -\tau_{rz2}^{(1)}, \qquad \tau_{rz0}^{(2)} = -\tau_{rz2}^{(2)}, \qquad \tau_{rz0}^{(s)} = -\tau_{rz2}^{(s)} - \tau_{rz}^{*\,(s)}\Big|_{\zeta=1} \quad (s > 2)$$

$$\tau_{\theta z0}^{(1)} = -\tau_{\theta z2}^{(1)}, \qquad \tau_{\theta z0}^{(2)} = -\tau_{\theta z2}^{(2)}, \qquad \tau_{\theta z0}^{(s)} = -\tau_{\theta z2}^{(s)} - \tau_{\theta z}^{*\,(s)}\Big|_{\zeta=1} \quad (s > 2)$$

From Formulas (1.4) and (1.5) it is clear that the quantities $Q_i^{(s)}$ are expressible in terms of the function $w_0^{(s)}$, which satisfies Equation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)^2 w_0^{(s)} = -\frac{4h^3}{D}\sigma_{z3}^{(s)}, \qquad D = \frac{2Eh^3}{3(1-\nu^2)} \tag{1.8}$$

2. The states of stress which can be made to decay rapidly as desired with distance from the edge for sufficiently small h are determined in [1] with the help of the first and second variants of the auxiliary iteration process.

For the first variant, q in Formula (1.1) takes the following values:

$$q = \lambda \quad \text{for } \tau_{r\theta}, \tau_{\theta z}, \qquad q = \lambda - 1 \quad \text{for } \sigma_r, \sigma_\theta, \tau_{r\theta}, \tau_{rz}$$
$$q = \lambda - 2 \quad \text{for } u, w, \qquad q = \lambda - 1 \quad \text{for } v \tag{2.1}$$

The system of equations for $Q_{(1)}^{(s)}$ in this case takes the form

$$\frac{\partial \sigma_r^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \tau_{r\theta}^{(s)}}{\partial \theta} + \frac{\partial \tau_{rz}^{(s)}}{\partial \zeta} + \frac{\sigma_r^{(s-1)} - \sigma_\theta^{(s-1)}}{r} = 0, \qquad E\frac{\partial u^{(s)}}{\partial \rho} = \sigma_r^{(s)} - \nu(\sigma_\theta^{(s)} + \sigma_z^{(s)})$$

$$\frac{\partial \tau_{rz}^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \tau_{\theta z}^{(s)}}{\partial \theta} + \frac{\partial \sigma_z^{(s)}}{\partial \zeta} + \frac{\tau_{rz}^{(s-1)}}{r} = 0, \qquad E\frac{\partial w^{(s)}}{\partial \zeta} = \sigma_z^{(s)} - \nu(\sigma_r^{(s)} + \sigma_\theta^{(s)}) \tag{2.2}$$

$$E\left(\frac{1}{r}\frac{\partial v^{(s)}}{\partial \theta} + \frac{u^{(s-1)}}{r}\right) = \sigma_\theta^{(s)} - \nu(\sigma_r^{(s)} + \sigma_z^{(s)}), \qquad E\left(\frac{\partial w^{(s)}}{\partial \rho} + \frac{\partial u^{(s)}}{\partial \zeta}\right) = 2(1+\nu)\tau_{rz}^{(s)}$$

$$\frac{\partial \tau_{r\theta}^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \sigma_\theta^{(s-2)}}{\partial \theta} + \frac{\partial \tau_{\theta z}^{(s)}}{\partial \zeta} + \frac{2\tau_{r\theta}^{(s-1)}}{r} = 0, \qquad E\left(\frac{\partial v^{(s)}}{\partial \zeta} + \frac{1}{r}\frac{\partial w^{(s-2)}}{\partial \theta}\right) = 2(1+\nu)\tau_{\theta z}^{(s)}$$

$$E\left(\frac{1}{r}\frac{\partial u^{(s-2)}}{\partial \theta} + \frac{\partial v^{(s)}}{\partial \rho} - \frac{v^{(s-1)}}{r}\right) = 2(1+\nu)\tau_{r\theta}^{(s)} \tag{2.3}$$

Here and in the sequel the variables ρ and ζ are introduced by the substitution

$$\frac{\partial}{\partial z} = h^{-1}\frac{\partial}{\partial \zeta}, \qquad \frac{\partial}{\partial r} = h^{-1}\frac{\partial}{\partial \rho}$$

The rate of change of all the stresses and displacements with respect to the variables ρ, θ and ζ is assumed to be of the same order. The quantity r entering in the equations is defined by the equality $r = R - \delta h$, where $0 \leqslant \delta \leqslant 1$.

The solution of the system (2.2) and (2.3) is also obtained in the form $Q_{(1)}^{(s)} = Q_{(1)}^{[s]} + Q_{(1)}^{*\,(s)}$, where $Q_{(1)}^{[s]}$ and $Q_{(1)}^{*\,(s)}$ have the same meaning as in the basic iteration process.

It is easily seen that in cylindrical coordinates only $Q_{(1)}^{*\,(1)}$ will be zero (in Cartesian coordinates $Q_{(1)}^{*(1)}$ and $Q_{(1)}^{*(2)}$ are identically zero)

The equations satisfied by $Q_{(1)}^{[s]}$, break down into two systems, of which the basic one is the homogeneous system corresponding to (2.3). The equations of this basic system coincide with the equations of torsion of a prismatic rod.

It is easily shown that the quantities $Q_{(1)}^{[s]}$ are expressible in terms of the harmonic function $\Psi(\rho,\zeta)$ in the following manner:

$$\tau_{r\theta\,(1)}^{[s]} = \frac{\partial \Psi^{(s)}}{\partial \rho}, \qquad \tau_{\theta z\,(1)}^{[s]} = \frac{\partial \Psi^{(s)}}{\partial \zeta}, \qquad Ev_{(1)}^{[s]} = 2\,(1+\nu)\,\Psi_{\zeta}^{(s)}$$

$$Eu_{(1)}^{[s]} = -2\,(1+\nu)\int \frac{1}{r}\frac{\partial \Psi^{(s)}}{\partial \theta}\,d\rho, \qquad \tau_{rz\,(1)}^{[s]} = -\int \frac{1}{r}\frac{\partial^2 \Psi^{(s)}}{\partial \zeta \partial \theta}\,d\rho \tag{2.4}$$

$$\sigma_{r\,(1)}^{[s]} = -2\,\frac{1}{r}\frac{\partial \Psi^{(s)}}{\partial \theta}, \quad \sigma_{\theta\,(1)}^{[s]} = 2\,\frac{1}{r}\frac{\partial \Psi^{(s)}}{\partial \theta} \quad Ew_{(1)}^{[s]} = 0, \quad \sigma_{z\,(1)}^{[s]} = 0$$

For the second variant of the auxiliary iteration process, q in (1.1) takes the values

$$q = \mu - 1 \quad \text{for } \sigma_r, \sigma_\theta, \sigma_z, \tau_{rz}, \qquad q = \mu - 2 \quad \text{for } \tau_{r\theta}, \tau_{\theta z}$$
$$q = \mu - 2 \quad \text{for } u, w, \qquad q = \mu - 3 \quad \text{for } v \tag{2.5}$$

The system of equations for $Q_{(2)}^{(s)}$ are

$$\frac{\partial \sigma_r^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \tau_{r\theta}^{(s-2)}}{\partial \theta} + \frac{\partial \tau_{rz}^{(s)}}{\partial \zeta} + \frac{\sigma_z^{(s-1)} - \sigma_\theta^{(s-1)}}{r} = 0$$
$$\frac{\partial \tau_{rz}^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \tau_{\theta z}^{(s-2)}}{\partial \theta} + \frac{\partial \sigma_z^{(s)}}{\partial \zeta} + \frac{\tau_{rz}^{(s-1)}}{r} = 0 \tag{2.6}$$

$$E\,\frac{\partial u^{(s)}}{\partial \rho} = \sigma_r^{(s)} - \nu\,(\sigma_\theta^{(s)} + \sigma_z^{(s)}), \qquad E\left(\frac{1}{r}\frac{\partial v^{(s-2)}}{\partial \theta} + \frac{u^{(s-1)}}{r}\right) = \sigma_\theta^{(s)} - \nu\,(\sigma_r^{(s)} + \sigma_z^{(s)})$$

$$E\,\frac{\partial w^{(s)}}{\partial \zeta} = \sigma_z^{(s)} - \nu\,(\sigma_r^{(s)} + \sigma_\theta^{(s)}), \qquad E\left(\frac{\partial w^{(s)}}{\partial \rho} + \frac{\partial u^{(s)}}{\partial \zeta}\right) = 2\,(1+\nu)\,\tau_{rz}^{(s)}$$

$$\frac{\partial \tau_{r\theta}^{(s)}}{\partial \rho} + \frac{1}{r}\frac{\partial \sigma_\theta^{(s)}}{\partial \theta} + \frac{\partial \tau_{\theta z}^{(s)}}{\partial \zeta} + \frac{2\tau_{r\theta}^{(s-1)}}{r} = 0$$

$$E\left(\frac{1}{r}\frac{\partial u^{(s)}}{\partial \theta} + \frac{\partial v^{(s)}}{\partial \rho} - \frac{v^{(s-1)}}{r}\right) = 2\,(1+\nu)\,\tau_{r\theta}^{(s)}, \quad E\left(\frac{\partial v^{(s)}}{\partial \zeta} + \frac{1}{r}\frac{\partial w^{(s)}}{\partial \theta}\right) = 2\,(1+\nu)\,\tau_{\theta z}^{(s)} \tag{2.7}$$

The general solution of this system has the form $Q_{(2)}^{(s)} = Q_{(2)}^{[s]} + Q_{(2)}^{*(s)}$ and here $Q_{(2)}^{*(1)} \equiv 0$ and $Q_{(2)}^{*(s)}$ differs from zero for $s \geqslant 2$.

The homogeneous system, which we obtain by setting the quantities in Equations (2.6) and (2.7) with superscripts $(s-1)$ and $(s-2)$ equal to zero, breaks into two systems, of which the basic system is the one corresponding to (2.6). The equations of the basic system coincide with the equations of the problem of plane deformation.

The stresses and displacements $Q_{(2)}^{[s]}$ are expressed in terms of the biharmonic function $\Phi^{(s)}(\rho, \zeta)$.

3. The determination of the state of bending stress in the plate reduces to the successive (in order of increasing s) determination of the functions $w_0^{(s)}$, $\Psi^{(s)}$ and $\Phi^{(s)}$

The boundary conditions for these functions were introduced in [1] in Cartesian coordinates; in cylindrical coordinates they differ only by the presence of terms with $Q_{(1)}^{*(2)}$ and $Q_{(2)}^{*(2)}$.

In the case of a plate with free edges, one takes $\lambda = 2$, $\mu = 2$ in

Formulas (2.1) and (2.5).

For the determination of the function $w_0^{(s)}$ we have Equation (1.8), in which $\sigma_{z3}^{(s)}$ on the right-hand side is determined by Formula (1.7), in particular

$$\sigma_{z3}^{(1)} = -{}^1/_4 P, \qquad \sigma_{z3}^{(2)} = 0, \qquad \sigma_{z3}^{(3)} = \frac{8-3\nu}{40(1-\nu)}\Delta p \tag{2.8}$$

The boundary conditions for the first three approximations for $r = R$ take the form

$$\sigma_{r1}^{(1)} = 0, \quad \sigma_{r1}^{(2)} = -\frac{3}{2}\int_{-1}^{1}\zeta\sigma_{r\,(1)}^{[1]}d\zeta, \; \sigma_{r1}^{(3)} = -\frac{3}{2}\int_{-1}^{1}\zeta\,(\sigma_r^{*\,(3)} + \sigma_{r\,(1)}^{[2]} + \sigma_{r\,(1)}^{*\,(2)} + \sigma_{r\,(2)}^{*(2)})\,d\zeta$$

$$\frac{1}{r}\frac{\partial\tau_{r\theta1}^{(1)}}{\partial\theta} - 2\tau_{rz2}^{(1)} = 0 \tag{2.9}$$

$$\frac{1}{r}\frac{\partial\tau_{r\theta1}^{(2)}}{\partial\theta} - 2\tau_{rz2}^{(2)} = \frac{3}{2}\int_{-1}^{1}d\zeta\int_{-1}^{\zeta}\frac{1}{r}\frac{\partial\tau_{r\theta\,(1)}^{*\,(2)}}{\partial\theta}d\zeta - \frac{3}{2}\int_{-1}^{1}(\tau_{rz\,(1)}^{*\,(2)} + \tau_{rz\,(2)}^{*\,(2)})\,d\zeta$$

$$\frac{1}{r}\frac{\partial\tau_{r\theta1}^{(3)}}{\partial\theta} - 2\tau_{rz2}^{(3)} = \frac{3}{2}\int_{-1}^{1}d\zeta\int_{-1}^{\zeta}\frac{1}{r}\frac{\partial}{\partial\theta}(\tau_{r\theta\,(1)}^{*\,(3)} + \tau_{r\theta}^{*\,(3)} + \tau_{r\theta\,(2)}^{[1]})\,d\zeta -$$

$$-\frac{3}{2}\int_{-1}^{1}(\tau_{rz}^{*\,(3)} + \tau_{rz\,(1)}^{*\,(3)} + \tau_{rz\,(2)}^{*\,(3)})\,d\zeta$$

The function $\Psi^{(s)}$ is harmonic, i.e. $\Delta\Psi^{(s)}(\rho,\zeta) = 0$ in the region $\rho \leqslant R/h$, $\zeta = \pm 1$.

As $\rho \to 0$ the function $\Psi^{(s)}$ together with its derivatives up to the second order must vanish, this ensures the decay of this state of stress with distance inward from the edge of the plate.

The boundary conditions for $\Psi^{(s)}$, in particular for $\Psi^{(1)}$ and $\Psi^{(2)}$, have the form

$$\partial\Psi^{(s)}/\partial\zeta = 0 \qquad \text{for } \zeta = \pm 1$$

$$\tau_{r\theta\,(1)}^{[1]} = -\zeta\tau_{r\theta1}^{(1)}, \qquad \tau_{r\theta\,(1)}^{[2]} = -\zeta\tau_{r\theta1}^{(2)} - \tau_{r\theta\,(1)}^{*\,(2)} \qquad \text{for } \rho = R/h \tag{2.10}$$

The function $\Phi^{(s)}$ is biharmonic, i.e. $\Delta\Delta\Phi^{(s)}(\rho,\zeta) = 0$ in the region $\zeta = \pm 1$, $\rho \leqslant R/h$, and it must vanish together with its derivatives up to third order for $\rho \to 0$.

The boundary conditions for $\Phi^{(s)}$, in particular, for $\Phi^{(1)}$ and $\Phi^{(2)}$, have the form

$$\text{for } \zeta = \pm 1 \quad \Phi^{(s)} = 0, \qquad \partial\Phi^{(s)}/\partial\zeta = 0$$

$$\text{for } \rho = R/h \qquad \sigma_{r\,(2)}^{[1]} = -\sigma_{r\,(1)}^{[1]} + \frac{3}{2}\zeta\int_{-1}^{1}\zeta\sigma_{r\,(1)}^{[1]}d\zeta, \qquad \frac{d\tau_{rz\,(2)}^{[1]}}{\partial\zeta} = 0$$

$$\sigma_{r\,(2)}^{[2]} = -\sigma_{r\,(1)}^{[2]} - \sigma_{r\,(1)}^{*\,(2)} - \sigma_{r\,(2)}^{*\,(2)} - \sigma_r^{*\,(3)} - \frac{3}{2}\zeta\int_{-1}^{1}\zeta\,(\sigma_r^{*\,(3)} + \sigma_{r\,(1)}^{[2]} + \sigma_{r\,(1)}^{*\,(2)} + \sigma_{r\,(2)}^{*(2)})\,d\zeta$$

$$\frac{\partial\tau_{rz\,(2)}^{[2]}}{\partial\zeta} = -\frac{\partial\tau_{rz\,(2)}^{*\,(2)}}{\partial\zeta} + \zeta\left(\frac{1}{r}\frac{\partial\tau_{r\theta1}^{(2)}}{\partial\theta} - 2\tau_{rz2}^{(2)}\right) + \frac{1}{r}\frac{\partial\tau_{r\theta\,(1)}^{*\,(2)}}{\partial\theta} - \frac{\partial\tau_{rz\,(1)}^{*\,(2)}}{\partial\zeta}$$

We note that in the axisymmetric case of bending of a plate, we have homogeneous equations in the functions $\Psi^{(1)}$, $\Phi^{(1)}$, $w_0^{(2)}$ and $\Psi^{(2)}$ with homogeneous boundary conditions, hence in this case these functions are identically zero.

4. We will call the rate of growth or decay (with decreasing h) of the stresses and displacements their order. Then it follows from (1.2) that the order of the stresses and displacements determined by the function

$$w^{(s)} = h^{s-4} w_0{}^{(s)},$$

is

$$\sigma_r, \sigma_\theta, \tau_{r\theta} \sim h^{s-3}, \quad \tau_{rz}, \tau_{\theta z} \sim h^{s-2}, \quad \sigma_z \sim h^{s-1}, \quad u, v \sim h^{s-3}, \quad w \sim h^{s-4} \tag{4.1}$$

while the quantities associated with $\Psi^{(s)}$, are from (2.1)

$$\tau_{r\theta}, \tau_{\theta z} \sim h^{s-3}, \quad \sigma_r, \sigma_\theta, \sigma_z, \tau_{rz} \sim h^{s-2}, \quad u, w \sim h^{s-1}, \quad v \sim h^{s-2} \tag{4.2}$$

and it is clear from (2.5) that the quantities related to $\Phi^{(s)}$, are

$$\sigma_r, \sigma_\theta, \sigma_z, \tau_{rz} \sim h^{s-2}, \quad \tau_{r\theta}, \tau_{\theta z} \sim h^{s-1}, \quad u, w \sim h^{s-1}; \quad v \sim h^{s} \tag{4.3}$$

The problem of deriving various approximate theories of bending of a plate may be treated as a problem of carrying out some number of approximations in the basic and auxiliary iteration processes. Thus, the classical theory may be considered as the problem of forming the first approximation of the basic iteration process.

The basic state of stress determines more or less accurately the stresses far from the edges of the plate. For $s = 1$ it follows from (4.1) that far from the edges of the plate the main stresses are σ_r, σ_θ and $\tau_{r\theta}$; the stresses $\tau_{rz}, \tau_{\theta z}$ and particularly σ_z are secondary stresses.

For the free edge of a plate (here we consider only the free edge), there are superimposed on the basic state of stress additional states of stress which decay very rapidly (for small h) with distance from the edge. The order of these edge stresses can be seen from Formulas (4.2) and (4.3). For $s = 1$ it follows from (4.2) that the corrections obtained at the edge in the principal stresses of the basic state of stress are relatively small for σ_r and σ_θ but the correction in $\tau_{r\theta}$ is of the same order as $\tau_{r\theta}$ itself in the basic state of stress. Moreover, the stress $\tau_{\theta z}$, attains the same order at the edge as the principal stresses in the basic state of stress. (According to classical theory even the order of $\tau_{\theta z}$ at the edge cannot be determined correctly). We note that the presence on the free edge of stresses $\tau_{r\theta}$ and $\tau_{\theta z}$ of the same order as the principal stresses determined by classical theory is due to the twisting of the edge.

From the preceding analysis which refers only to the case of a free edge, it follows that with the help of the classical theory of plates, the state of stress may be determined in the first approximation both far from the edge and on the edge only in the axisymmetric case of bending, where there is no twisting of the edge. In the nonaxisymmetric case it is necessary to construct the functions $w_0{}^{(1)}$ and $\psi^{(1)}$ in order to determine in the first approximation the state of stress both far from the edge and at the edge. The stresses $\sigma_r, \sigma_\theta, \tau_{r\theta}$ and $\tau_{\theta z}$ will be the quantities of order h^{-3}, while $\tau_{rz} \sim h^{-1}$, $\sigma_z \sim h^0$.

The first refinement, i.e. retaining terms of order h in comparison to terms of order h^0, is determined by the function $w_0{}^{(2)}$; far from the edge. In order to obtain a refinement of the same order in the edge stresses, it is necessary to construct the functions $\Phi^{(1)}$ and $\Psi^{(2)}$ as well.

In the axisymmetric case of bending, $w_0{}^{(2)} \equiv 0$, hence the first refinement in the stresses far from the edge is determined by the function $w_0{}^{(3)}$ and for σ_r, σ_θ and $\tau_{r\theta}$ will be of order h^0.

We will consider some examples, replacing $w_0{}^{(s)}$ everywhere in the following by the function $w^{(s)} = h^{s-4} w_0{}^{(s)}$, the meaning of

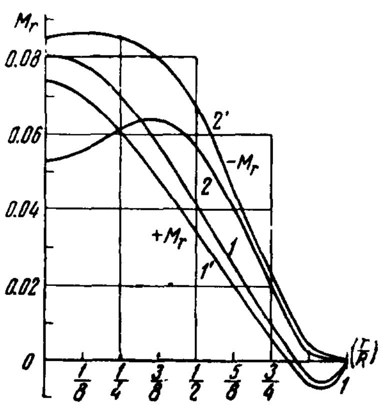

Fig. 1

which is clear from the series (1.1) for w in the basic iteration process.

5. A circular plate of radius R and thickness $2h$ with a free edge is bent under the action of an axisymmetric load of intensity

$$p = p_0 \cos nr$$

The load will be self-equilibrating if n satisfies Equation

$$n = \frac{2k\pi}{R} \quad \text{or} \quad \tan\frac{nR}{2} = nR \quad (k = 1, 2, 3, \ldots) \tag{5.1}$$

We will refine the values of the stress only far from the edge, for which we need construct only the functions $w^{(1)}$ and $w^{(3)}$

For $w^{(1)}$ in the axisymmetric case we have

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w^{(1)} = \frac{P_0}{D}\cos nr, \qquad \sigma_{r1}^{(1)} = 0, \qquad \tau_{rz2}^{(1)} = 0 \qquad \text{for } r = R$$

whence

$$w^{(1)} = -\frac{P_0 r^2}{8Dn^2(1+\nu)}\left[-4\cos nR - (3+\nu) + \frac{6(1-\nu)}{nR}\sin nR - \frac{6(1-\nu)}{n^2R^2}(1-\cos nR) - \right.$$
$$\left. - 2(1+\nu)F(R)\right] + \frac{P_0}{4Dn^2}\left[-\frac{r}{n}\sin nr + \frac{5}{n^2}\cos nr + \left(\frac{6}{n^2} - r^2\right)F(r)\right] \left(F(r) = \int\frac{1-\cos nr}{r}\,dr\right)$$

For $w^{(3)}$ we obtain Equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)^2 w^{(3)} = \frac{1}{D}\frac{(8-3\nu)P_0h^2}{10(1-\nu)}\left(n^2\cos nr + \frac{n\sin nr}{r}\right)$$

$$\sigma_{r1}^{(3)} = \frac{3(8-3\nu)P_0}{20(1-\nu)}\cos nR - \frac{3(8+\nu)P_0}{20}\left[\frac{1}{nR}\sin nR - \frac{1-\cos nR}{n^2R^2}\right] \qquad \text{for } r = R$$

whence

$$w^{(3)} = -\frac{P_0h^2r^2}{40D(1-\nu^2)}\left[-(3+\nu)(8-3\nu) - 2(1+\nu)(8-3\nu)F(R) + \right.$$
$$\left. + \frac{2(1-\nu)(8-7\nu)}{nR}\sin nR - \frac{2(1-\nu)(8-7\nu)}{n^2R^2}(1-\cos nR)\right] +$$
$$+ \frac{P_0h^2(8-3\nu)}{40D(1-\nu)}\left[\frac{2r}{n}\sin nr + \frac{6}{n^2}\cos nr - \left(2r^2 - \frac{8}{n^2}\right)F(r)\right]$$

From the graphs (Fig.1) of bending moments $M_r^{(1)}$ (1 and 2) and $M_r^{(1)} + M_r^{(3)}$ (1′and 2′) for the cases of the smallest roots of Equations (5.1) and $h/R = 0.1$ it is clear that for $n_1 = 2.332R^{-1}$, where the rate of change of the load along a radius is not very large, the corrections determined by the function $w^{(3)}$, will not be very significant. In the case $n_2 = 2\pi/R$ these corrections turn out to be very significant: even for $h/R = 0.1$ they are commensurate with the values obtained from classical theory.

It should be noted that for some forms of loading the corrections determined by the function $w^{(3)}$, will be very small. Thus in particular the corrections for a self-equilibrating parabolic load do not exceed 1% . From this one may conclude that there exist loads for which the classical theory is sufficiently accurate

For an example of a nonaxisymmetric bending we consider the problem of a stress concentration

We will consider the cylindrical bending of an infinite plate with a circular hole of radius R

We represent the stresses in the plate in the form $Q = Q^{(0)} + Q^{(c)}$, where $Q^{(c)}$ are the stresses in a solid plate under the action of bending moments $M_x = M$ and $M_y = 0$ applied at infinity, and $Q^{(0)}$ are the stresses in a plate containing a hole, which is acted upon by a system of forces applied on its contour.

As is known [2 and 3], in a solid infinite plate in cylindrical bending, the state of stress at the section corresponding to the contour of the hole is

$$M_r^{(c)} = + {}^1/_2\, M\,(1 + \cos 2\theta), \qquad M_\theta^{(c)} = {}^1/_2\, M\,(1 - \cos 2\theta), \qquad H_{r\theta}^{(c)} = -{}^1/_2\, M \sin 2\theta$$

$$V_r^{(c)} = Q_r^{(c)} + \frac{1}{r}\frac{\partial H_{r\theta}^{(c)}}{\partial\theta} = -MR^{-1}\cos 2\theta, \qquad Q_r^{(c)} = 0, \qquad Q_\theta^{(c)} = 0 \qquad (4.4)$$

The quantities $Q^{(0)}$ are determined by the above stated method. Since the problem is to redistribute the stresses on the contour of the hole, we will construct only the functions $w^{(1)}$ and $\Psi^{(1)}$. In the case under consideration the stresses applied on the contour of the hole have the form

$$\sigma_r = -{}^3/_4\, Mh^{-2}\,\zeta\,(1 + \cos 2\theta), \qquad \tau_{r\theta} = {}^3/_4\, Mh^{-2}\,\zeta \sin 2\theta, \qquad \tau_{rz} = 0$$

We have the homogeneous equation $\Delta\Delta w^{(1)} = 0$ for the function $w^{(1)}$ with the inhomogeneous boundary conditions

$$\sigma_{r1}^{(1)} = -{}^3/_4\, M\,(1 + \cos 2\theta), \qquad \frac{1}{r}\frac{\partial\tau_{r\theta 1}^{(1)}}{\partial\theta} - 2\tau_{rz2}^{(1)} = MR^{-1}\cos 2\theta \qquad \text{for } r = R$$

Hence

$$w^{(1)} = -\frac{MR^2}{2D}\left[\frac{1}{1-\nu}\ln r + \frac{1}{3+\nu}\left(1 - \frac{R^2}{2r^2}\right)\cos 2\theta\right]$$

On the contour of the hole

$$M_\theta^{(1)} = \frac{M}{2}\left[1 - \frac{1+3\nu}{3+\nu}\cos 2\theta\right], \qquad H_{r\theta}^{(1)} = -\frac{M\,(1-\nu)}{2\,(3+\nu)}\sin 2\theta$$

$$Q_\theta^{(1)} = \frac{4M}{(3+\nu)\,R}\sin 2\theta$$

For the function $\Psi^{(1)}$ we have the harmonic equation $\Delta\Psi^{(1)}(\rho,\zeta) = 0$ with the boundary conditions

$$\frac{\partial\Psi^{(1)}}{\partial\zeta} = 0 \quad \text{for } \zeta = \pm 1, \qquad \tau_{r\theta(1)}^{[1]} = \zeta\left[\frac{3M}{4} + \frac{3M\,(1-\nu)}{4\,(3+\nu)}\right]\sin 2\theta \qquad \text{for } \rho = R/h$$

and we obtain $\Psi^{(1)}$ in the form

$$\Psi^{(1)} = \frac{48M\sin 2\theta}{(3+\nu)\pi^3}\sum_{s=1}^{\infty}\frac{(-1)^s}{(2s-1)^3}\exp\left[\frac{(2s-1)\pi}{2}\left(\frac{R}{h} - \rho\right)\right]\sin\frac{(2s-1)\pi}{2}\zeta$$

On the contour of the hole

$$M_{\theta(1)}^{(1)} = -\frac{5.07M\cos 2\theta}{3+\nu}\frac{h}{R}, \qquad H_{r\theta(1)}^{(1)} = \frac{2M\sin 2\theta}{3+\nu}, \qquad Q_{\theta(1)}^{(1)} = -\frac{3.38\,Mh^{-1}}{3+\nu}\sin 2\theta$$

In this manner we obtain on the contour of the hole

$$M_\theta = M_\theta^{(c)} + M_\theta^{(1)} + M_{\theta(1)}^{(1)} = M\left\{1 - \frac{1}{3+\nu}\left[2\,(1+\nu) + 5.07\,\frac{h}{R}\right]\cos 2\theta\right\}$$

$$H_{r\theta} = H_{r\theta}^{(c)} + H_{r\theta}^{(1)} + H_{r\theta(1)}^{(1)} = 0$$

$$Q_\theta = Q_\theta^{(c)} + Q_\theta^{(1)} + Q_{\theta(1)}^{(1)} = \frac{Mh^{-1}}{3+\nu}\left[4\frac{h}{R} - 3.38\right]\sin 2\theta$$

For $\nu = 0.3$, $R/h = 10$, $\theta = {}^1/_2\pi$ we obtain $\max M_\theta = 1.94\, M$, $k = 1.94$.

For $\nu = 0.3$, $R/h = 3$, $\theta = {}^1/_2\pi$ we obtain $\max M_\theta = 2.3\, M$, $k = 2.3$.

According to the calssical theory the coefficient of stress concentration in this case is $k = 1.79$, independent of R/h. Consequently, if the stress concentration is determined only on M_θ, one may assert that the result of classical theory will be relatively accurate only for $R/h > 10$. However, it is clear from the expression for Q_θ, that the transverse tangential stresses $\tau_{\theta z}$ will not be negligibly small and should not be neglected.

For small R/h the stress concentration coefficient grows and the classical result becomes very inaccurate. Thus, even for $R/h = 3$ the error in determining the stress concentration coefficient is about 25% (from Reissner's theory, it is upwards of 10%) [4]. However, for small R/h ($R/h < 1$), the problem becomes essentially three-dimensional, and none of the approximate methods, including that of the present article, may be used.

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